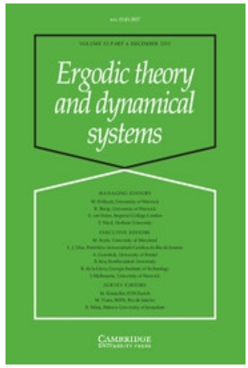
Parameter rigid actions of simply connected nilpotent Lie groups

MARUHASHI, HIROKAZU

Ergodic Theory and Dynamical Systems

http://journals.cambridge.org/ETS



Parameter rigid actions of simply connected nilpotent Lie groups

HIROKAZU MARUHASHI

Ergodic Theory and Dynamical Systems / Volume 33 / Issue 06 / December 2013, pp 1864 - 1875
DOI: 10.1017/etds.2012.112, Published online: 22 August 2012

Link to this article: http://journals.cambridge.org/abstract_S0143385712001125

How to cite this article:
HIROKAZU MARUHASHI (2013). Parameter rigid actions of simply connected nilpotent Lie groups. Ergodic Theory and Dynamical Systems, 33, pp 1864-1875 doi:10.1017/etds.2012.112

Ergod. Th. & Dynam. Sys. (2013), **33**, 1864–1875 © *Cambridge University Press, 2012*
doi:10.1017/etds.2012.112

Parameter rigid actions of simply connected nilpotent Lie groups

HIROKAZU MARUHASHI†

Department of Mathematics, Kyoto University, Japan
(e-mail: h-maruha@math.kyoto-u.ac.jp)

(*Received* 5 *May* 2011 *and accepted in revised form* 19 *April* 2012)

Abstract. We show that for a locally free C^∞-action of a connected and simply connected nilpotent Lie group on a compact manifold, if every real-valued cocycle is cohomologous to a constant cocycle, then the action is parameter rigid. The converse is true if the action has a dense orbit. Using this, we construct parameter rigid actions of simply connected nilpotent Lie groups whose Lie algebras admit rational structures with graduations. This generalizes the results of dos Santos [Parameter rigid actions of the Heisenberg groups. *Ergod. Th. & Dynam. Sys.* **27** (2007), 1719–1735] concerning the Heisenberg groups.

1. *Introduction*

Let G be a connected Lie group with Lie algebra $\mathfrak{g}$ and M a C^∞-manifold without boundary. Let $\rho : M \times G \to M$ be a C^∞ right action. We call ρ *locally free* if every isotropy subgroup of ρ is discrete in G. Assume that ρ is locally free. Then we have the orbit foliation $\mathcal{F}$ of ρ whose tangent bundle $T\mathcal{F}$ is naturally isomorphic to a trivial bundle $M \times \mathfrak{g}$.

The action ρ is *parameter rigid* if any action ρ' of G on M with the same orbit foliation $\mathcal{F}$ is C^∞-conjugate to ρ; more precisely, there exist an automorphism Φ of G and a C^∞-diffeomorphism F of M which preserves each leaf of $\mathcal{F}$ and is homotopic to the identity through C^∞-maps preserving each leaf of $\mathcal{F}$ such that

$$F(\rho(x, g)) = \rho'(F(x), \Phi(g))$$

for all $x \in M$ and $g \in G$.

Parameter rigidity of actions has been studied by several authors, for instance, Katok and Spatzier [**3**], Matsumoto and Mitsumatsu [**4**], Mieczkowski [**5**], Ramírez [**7**] and dos Santos [**8**]. Most of the known examples of parameter rigid actions are those of abelian groups, and non-abelian actions have not been considered so much.

† Research Fellow of the Japan Society for the Promotion of Science.

Parameter rigidity is closely related to cocycles over actions. Now suppose G is contractible and M is compact. Let H be a Lie group. A C^∞-map $c : M \times G \to H$ is called a *H-valued cocycle over ρ* if c satisfies

$$c(x, gg') = c(x, g)c(\rho(x, g), g')$$

for all $x \in M$ and $g, g' \in G$.

A cocycle c is *constant* if $c(x, g)$ is independent of x. A constant cocycle is just a homomorphism $G \to H$.

H-valued cocycles c, c' are *cohomologous* if there exists a C^∞-map $P : M \to H$ such that

$$c(x, g) = P(x)^{-1}c'(x, g)P(\rho(x, g))$$

for all $x \in M$ and $g \in G$.

The action ρ is *H-valued cocycle rigid* if every H-valued cocycle over ρ is cohomologous to a constant cocycle.

PROPOSITION 1. [**4**] *If ρ is G-valued cocycle rigid, then it is parameter rigid.*

Remark. In [**4**] Matsumoto and Mitsumatsu assume that ρ has at least one trivial isotropy subgroup, but this assumption is not necessary.

PROPOSITION 2. [**4**] *When $G = \mathbb{R}^n$, the following are equivalent:*
(1) *ρ is $\mathbb{R}$-valued cocycle rigid;*
(2) *ρ is $\mathbb{R}^n$-valued cocycle rigid;*
(3) *ρ is parameter rigid.*

Remark. The equivalence of the first two conditions is obvious.

In this paper we consider actions of simply connected nilpotent Lie groups. In [**8**], dos Santos proved that, for actions of a Heisenberg group H_n, $\mathbb{R}$-valued cocycle rigidity implies H_n-valued cocycle rigidity, and, using this, he constructed parameter rigid actions of Heisenberg groups. To the best of my knowledge these are the only known non-trivial parameter rigid actions of non-abelian nilpotent Lie groups. We prove the following.

THEOREM 1. *Let N be a connected and simply connected nilpotent Lie group, M a compact manifold and ρ a locally free C^∞-action of N on M. Then, the following are equivalent:*
(1) *ρ is $\mathbb{R}$-valued cocycle rigid;*
(2) *ρ is N-valued cocycle rigid;*
(3) *ρ is parameter rigid and every orbitwise constant real-valued C^∞-function of ρ on M is constant on M.*

This theorem enables us to construct parameter rigid actions of nilpotent Lie groups. The most interesting one is the following.

THEOREM 2. [**7**] *Let N denote the group of all upper triangular real matrices with* 1 *on the diagonal, Γ a cocompact lattice of* $\mathrm{SL}(n, \mathbb{R})$ *and ρ the action of N on $\Gamma\backslash\mathrm{SL}(n, \mathbb{R})$ by right multiplication. If $n \geq 4$, ρ is $\mathbb{R}$-valued cocycle rigid.*

Remark. In [**7**], Ramírez proved more general theorems.

COROLLARY. *The above action ρ is parameter rigid.*

In §4 we construct parameter rigid actions of nilpotent groups using Theorem 1. It is a generalization of dos Santos' example. Let N be a connected and simply connected nilpotent Lie group and Γ, Λ be lattices in N. Consider the action of Λ on $\Gamma\backslash N$ by right multiplication and let $\tilde{\rho}$ be its suspended action of N.

THEOREM 3. *If Λ is Diophantine with respect to Γ, then the action $\tilde{\rho}$ of N is parameter rigid.*

For the definition of Diophantine lattices, see §4.

2. *Preliminaries*

Let G be a contractible Lie group with Lie algebra $\mathfrak{g}$, M a compact manifold and ρ a locally free action of G on M with orbit foliation $\mathcal{F}$. Let H be a Lie group with Lie algebra $\mathfrak{h}$. We denote by $\Omega^p(\mathcal{F}, \mathfrak{h})$ the set of all C^∞-sections of $\mathrm{Hom}(\bigwedge^p T\mathcal{F}, \mathfrak{h})$. The exterior derivative

$$d_{\mathcal{F}} : \Omega^p(\mathcal{F}, \mathfrak{h}) \to \Omega^{p+1}(\mathcal{F}, \mathfrak{h})$$

is defined since $T\mathcal{F}$ is integrable.

By differentiating, H-valued cocycles over ρ are in one-to-one correspondence with $\mathfrak{h}$-valued leafwise one-forms $\omega \in \Omega^1(\mathcal{F}, \mathfrak{h})$ such that

$$d_{\mathcal{F}}\omega + [\omega, \omega] = 0.$$

PROPOSITION 3. *Let c_1, c_2 be H-valued cocycles over ρ and let ω_1, ω_2 be corresponding differential forms. For a C^∞-map $P : M \to H$, the following are equivalent:*

(1) $c_1(x, g) = P(x)^{-1}c_2(x, g)P(\rho(x, g))$ *for all $x \in M$ and $g \in G$;*

(2) $\omega_1 = \mathrm{Ad}(P^{-1})\omega_2 + P^*\theta$ *where $\theta \in \Omega^1(H, \mathfrak{h})$ is the left Maurer–Cartan form on H.*

COROLLARY. [**4**] *The following are equivalent:*

(1) *ρ is G-valued cocycle rigid;*

(2) *for each $\omega \in \Omega^1(\mathcal{F}, \mathfrak{g})$ such that $d_{\mathcal{F}}\omega + [\omega, \omega] = 0$, there exist an endomorphism $\Phi : \mathfrak{g} \to \mathfrak{g}$ of the Lie algebra and a C^∞-map $P : M \to G$ such that*

$$\omega = \mathrm{Ad}(P^{-1})\Phi + P^*\theta.$$

Proposition 3 is obtained by examining the proof of [**4**, Corollary 2]. In this paper, we will identify a cocycle with its corresponding differential form.

Let us consider real-valued cocycles. A real-valued cocycle over ρ is given by $\omega \in \Omega^1(\mathcal{F}, \mathbb{R})$ satisfying $d_{\mathcal{F}}\omega = 0$. Two real-valued cocycles ω_1, ω_2 are cohomologous if and only if $\omega_1 = \omega_2 + d_{\mathcal{F}}P$ for some C^∞-function $P : M \to \mathbb{R}$. Leafwise cohomology $H^*(\mathcal{F})$ of $\mathcal{F}$ is the cohomology of the cochain complex $(\Omega^*(\mathcal{F}, \mathbb{R}), d_{\mathcal{F}})$. Thus $H^1(\mathcal{F})$ is the set of all equivalence classes of real-valued cocycles.

The identification $T\mathcal{F} \simeq M \times \mathfrak{g}$ induces a map $H^*(\mathfrak{g}) \to H^*(\mathcal{F})$ where $H^*(\mathfrak{g})$ is the cohomology of the Lie algebra $\mathfrak{g}$. By the compactness of M, this map is injective on $H^1(\mathfrak{g})$. Hence we identify $H^1(\mathfrak{g})$ with its image. Note that $H^1(\mathfrak{g})$ is the set of all equivalence classes of constant real-valued cocycles. Thus real-valued cocycle rigidity is equivalent to $H^1(\mathcal{F}) = H^1(\mathfrak{g})$.

Notice that $H^0(\mathcal{F})$ is the set of leafwise constant real-valued C^∞-functions of $\mathcal{F}$ on M and $H^0(\mathfrak{g})$ consists of constant functions on M. Therefore the equivalence of (1) and (3) in Theorem 1 can be stated as follows: $H^1(\mathcal{F}) = H^1(\mathfrak{n})$ if and only if ρ is parameter rigid and $H^0(\mathcal{F}) = H^0(\mathfrak{n})$.

3. *Proof of Theorem 1*

Let N be a simply connected nilpotent Lie group with Lie algebra $\mathfrak{n}$, M a compact manifold and ρ a locally free action of N on M with orbit foliation $\mathcal{F}$.

We first prove that N-valued cocycle rigidity implies real-valued cocycle rigidity. There exist closed subgroups N' and A of N such that $N' \lhd N$, $N = N' \rtimes A$ and $A \simeq \mathbb{R}$. Let c be any real-valued cocycle over ρ. We regard c as an N-valued cocycle over ρ via the inclusion $\mathbb{R} \simeq A \hookrightarrow N$. By N-valued cocycle rigidity, there exist an endomorphism Φ of N and a C^∞-map $P : M \to N$ such that $c(x, g) = P(x)^{-1}\Phi(g)P(\rho(x, g))$ for all $x \in M$ and $g \in N$. Applying the natural projection $\pi : N \to A \simeq \mathbb{R}$, we obtain $c(x, g) = (\pi \circ P)(x)^{-1}(\pi \circ \Phi)(g)(\pi \circ P)(\rho(x, g))$. Thus c is cohomologous to a constant cocycle $\pi \circ \Phi$.

Next we assume $H^1(\mathcal{F}) = H^1(\mathfrak{n})$ and prove N-valued cocycle rigidity. We need the following two lemmata.

LEMMA 1. *Let V be a finite dimensional real vector space. Assume that $\omega \in \Omega^1(\mathcal{F}, V)$ satisfies the equation $d_{\mathcal{F}}\omega = \varphi$, where $\varphi \in \mathrm{Hom}(\bigwedge^2 \mathfrak{n}, V)$ is a constant leafwise two-form. Then there exists a constant leafwise one-form $\psi \in \mathrm{Hom}(\mathfrak{n}, V)$ with $\varphi = d_{\mathcal{F}}\psi$.*

Proof. Since N is amenable, there exists an N-invariant Borel probability measure μ on M. Define $\psi \in \mathrm{Hom}(\mathfrak{n}, V)$ by

$$\psi(X) = \int_M \omega(X)\, d\mu,$$

where $X \in \mathfrak{n}$. Since $\varphi(X, Y) = X\omega(Y) - Y\omega(X) - \omega([X, Y])$ for all $X, Y \in \mathfrak{n}$, we obtain

$$\varphi(X, Y) = -\int_M \omega([X, Y])\, d\mu.$$

Thus

$$d_{\mathcal{F}}\psi(X, Y) = -\psi([X, Y]) = -\int_M \omega([X, Y])\, d\mu = \varphi(X, Y),$$

and hence $d_{\mathcal{F}}\psi = \varphi$. □

Set $\mathfrak{n}^1 = \mathfrak{n}$, $\mathfrak{n}^i = [\mathfrak{n}, \mathfrak{n}^{i-1}]$. Then $\mathfrak{n}^s \neq 0$, $\mathfrak{n}^{s+1} = 0$ for some s. For each $1 \leq i \leq s$, choose a subspace V_i with $\mathfrak{n}^i = V_i \oplus \mathfrak{n}^{i+1}$, so that $\mathfrak{n} = \bigoplus_{i=1}^s V_i$.

LEMMA 2. *Let $\omega \in \Omega^1(\mathcal{F}, \mathfrak{n})$ be such that $d_{\mathcal{F}}\omega + [\omega, \omega] = 0$. Decompose ω as*

$$\omega = \xi + \omega_k + \omega_{k+1},$$

where $\xi \in \Omega^1(\mathcal{F}, \bigoplus_{i=1}^{k-1} V_i)$, $\omega_k \in \Omega^1(\mathcal{F}, V_k)$ and $\omega_{k+1} \in \Omega^1(\mathcal{F}, \mathfrak{n}^{k+1})$. If ξ is constant, then there exists $\omega' \in \Omega^1(\mathcal{F}, \mathfrak{n})$ with $d_{\mathcal{F}}\omega' + [\omega', \omega'] = 0$ which iscohomologous to ω and *such that*

$$\omega' = \xi' + \omega'_{k+1},$$

where $\xi' \in \Omega^1(\mathcal{F}, \bigoplus_{i=1}^k V_i)$ *is constant and* $\omega'_{k+1} \in \Omega(\mathcal{F}, \mathfrak{n}^{k+1})$.

Proof. By the cocycle equation,

$$0 = d_{\mathcal{F}}\xi + d_{\mathcal{F}}\omega_k + d_{\mathcal{F}}\omega_{k+1} + [\xi, \xi] + \text{an element of } \Omega^2(\mathcal{F}, \mathfrak{n}^{k+1}).$$

Comparing this with the V_k component, we see that $d_{\mathcal{F}}\omega_k$ is constant. Hence, by Lemma 1, $d_{\mathcal{F}}\omega_k = d_{\mathcal{F}}\psi$ for some $\psi \in \mathrm{Hom}(\mathfrak{n}, V_k)$. Since we are assuming that $H^1(\mathcal{F}) = H^1(\mathfrak{n})$, there exists $\psi' \in \mathrm{Hom}(\mathfrak{n}, V_k)$ and a C^∞-map $h : M \to V_k$ such that

$$\omega_k = \psi + \psi' + d_{\mathcal{F}}h.$$

Put $P = e^h : M \to N$. Let $x \in M$ and $X \in T_x\mathcal{F}$. Choose a path $x(t)$ such that $X = (d/dt)x(t)|_{t=0}$. Let $\theta \in \Omega^1(N, \mathfrak{n})$ be the left Maurer–Cartan form on N. Then

$$\begin{aligned} P^*\theta(X) &= \frac{d}{dt} P(x)^{-1} P(x(t))\bigg|_{t=0} = \frac{d}{dt} e^{-h(x)} e^{h(x(t))}\bigg|_{t=0} \\ &= \frac{d}{dt} \exp(-h(x) + h(x(t)) + \text{an element of } \mathfrak{n}^{k+1})\bigg|_{t=0} \\ &= d_{\mathcal{F}}h(X) + \text{an element of } \mathfrak{n}^{k+1}. \end{aligned}$$

Thus $P^*\theta = d_{\mathcal{F}}h + \text{an element of } \Omega^1(\mathcal{F}, \mathfrak{n}^{k+1})$. Note that $\mathrm{Ad}(P^{-1}) = \exp \mathrm{ad}(-h)$ is the identity on $\bigoplus_{i=1}^k V_i$ and preserves $\mathfrak{n}^{k+1}$. Hence

$$\begin{aligned} \omega - P^*\theta &= \xi + \psi + \psi' + \text{an element of } \Omega^1(\mathcal{F}, \mathfrak{n}^{k+1}) \\ &= \mathrm{Ad}(P^{-1})(\xi + \psi + \psi' + \text{an element of } \Omega^1(\mathcal{F}, \mathfrak{n}^{k+1})). \end{aligned}$$ □

Let ω be any N-valued cocycle. Using Lemma 2, we can exchange ω for a cohomologous cocycle whose V_1-component is constant. Applying Lemma 2 repeatedly, we eventually get a constant cocycle cohomologous to ω. This proves N-valued cocycle rigidity.

Next we assume that ρ is parameter rigid and $H^0(\mathcal{F}) = H^0(\mathfrak{n})$. Let $\mathfrak{n}^i$ and V_i be as above. Note that $\mathfrak{n}^s$ is central in $\mathfrak{n}$. Fix a non-zero $Z \in \mathfrak{n}^s$.

Let $[\omega] \in H^1(\mathcal{F})$. Let ω_0 be the N-valued cocycle over ρ corresponding to the constant cocycle $\mathrm{id} : N \to N$. We call ω_0 *the canonical one-form of* ρ. Fix an $\epsilon > 0$ and put $\eta := \omega_0 + \epsilon\omega Z$. Then η is an N-valued cocycle over ρ, since

$$d_{\mathcal{F}}\eta + [\eta, \eta] = d_{\mathcal{F}}\omega_0 + \epsilon(d_{\mathcal{F}}\omega)Z + [\omega_0, \omega_0] = 0.$$

Since M is compact, we can assume $\eta_x : T_x\mathcal{F} \to \mathfrak{n}$ is bijective for all $x \in M$ by choosing $\epsilon > 0$ small. There exists a unique action ρ' of N on M whose orbit foliation is $\mathcal{F}$ and canonical one-form is η. See [**1**]. By parameter rigidity, ρ' is conjugate to ρ. Thus there exist a C^∞-map $P : M \to N$ and an automorphism Φ of N satisfying

$$\omega_0 + \epsilon\omega Z = \mathrm{Ad}(P^{-1})\Phi_*\omega_0 + P^*\theta. \tag{1}$$

Note that $\log : N \to \mathfrak{n}$ is defined since N is simply connected and nilpotent. Let us decompose $\omega_0 = \sum_{i=1}^s \omega_{0i}$, $\Phi_*\omega_0 = \sum_{i=1}^s \omega'_{0i}$ and $\log P = \sum_{i=1}^s P_i$ according to the decomposition $\mathfrak{n} = \bigoplus_{i=1}^s V_i$.

LEMMA 3. *Assume that* $P_1 = \cdots = P_{k-1} = 0$; *that is,* $\log P \in \mathfrak{n}^k$.

(1) *If* $k < s$, *then there exist a* C^∞*-map* $Q : M \to N$ *and an automorphism* Ψ *of* N *such that*

$$\omega_0 + \epsilon\omega Z = \mathrm{Ad}(Q^{-1})\Psi_*\omega_0 + Q^*\theta$$

and $Q_1 = \cdots = Q_k = 0$, *where* $\log Q = \sum_{i=1}^s Q_i$.

(2) *If* $k = s$, *then* ω *is cohomologous to a constant cocycle.*

Proof. For all $X = (d/dt)x(t)|_{t=0} \in T_x\mathcal{F}$,

$$\begin{aligned} P^*\theta(X) &= \frac{d}{dt} P(x)^{-1}P(x(t))\bigg|_{t=0} = \frac{d}{dt}\exp\bigg(-\sum_{i=k}^{s} P_i(x)\bigg)\exp\bigg(\sum_{i=k}^{s} P_i(x(t))\bigg)\bigg|_{t=0} \\ &= \frac{d}{dt}\exp\bigg\{\sum_{i=k}^{s}(P_i(x(t)) - P_i(x)) + \text{an element of } \mathfrak{n}^{k+1}\bigg\}\bigg|_{t=0} \\ &= \frac{d}{dt}\exp(P_k(x(t)) - P_k(x) + \text{an element of } \mathfrak{n}^{k+1})\bigg|_{t=0} \\ &= d_{\mathcal{F}}P_k(X) + \text{an element of } \mathfrak{n}^{k+1}. \end{aligned}$$

We have

$$\begin{aligned} \mathrm{Ad}(P^{-1})\Phi_*\omega_0 &= \exp\bigg(\mathrm{ad}\bigg(-\sum_{i=k}^{s} P_i\bigg)\bigg)\sum_{i=1}^{s}\omega'_{0i} \\ &= \sum_{i=1}^{s}\omega'_{0i} + \text{an element of } \mathfrak{n}^{k+1}. \end{aligned}$$

Comparing this with the V_k-component of (1) we get

$$\omega_{0k} + \delta_{ks}\epsilon\omega Z = \omega'_{0k} + d_{\mathcal{F}}P_k.$$

When $k = s$, the equation

$$\omega Z = \epsilon^{-1}(\omega'_{0s} - \omega_{0s}) + d_{\mathcal{F}}(\epsilon^{-1}P_s)$$

shows that ω is cohomologous to a constant cocycle.

If $k < s$, then $d_{\mathcal{F}}P_k = \phi \circ \omega_0$ for some linear map $\phi : \mathfrak{n} \to V_k$. For any $X \in \mathfrak{n}$, let $\tilde{X}$ denote the vector field on M determined by X via ρ. We have $\tilde{X}P_k = \phi(X)$, and by integrating over an integral curve γ of $\tilde{X}$ we get $P_k(\gamma(T)) - P_k(\gamma(0)) = \phi(X)T$ for all $T > 0$. Since M is compact, $\phi(X) = 0$. Therefore $d_{\mathcal{F}}P_k = 0$, so P_k is constant on each leaf of $\mathcal{F}$. Thus P_k is constant on M by our assumption. Put $g := \exp(-P_k)$ and $Q := gP = \exp(\sum_{i=k+1}^{s} P_i + \text{an element of } \mathfrak{n}^{k+1})$. Then

$$\begin{aligned} \omega_0 + \epsilon\omega Z &= \mathrm{Ad}(Q^{-1}g)\Phi_*\omega_0 + (L_{g^{-1}} \circ Q)^*\theta \\ &= \mathrm{Ad}(Q^{-1})\Psi_*\omega_0 + Q^*\theta \end{aligned}$$

where $\Psi_* := \mathrm{Ad}(g)\Phi_*$. □

Applying Lemma 3 repeatedly, we see that ω is cohomologous to a constant cocycle.

Finally we assume $H^1(\mathcal{F}) = H^1(\mathfrak{n})$ and prove that ρ is parameter rigid and $H^0(\mathcal{F}) = H^0(\mathfrak{n})$. Parameter rigidity of ρ follows from Proposition 1. Let $f \in H^0(\mathcal{F})$. Fix a non-zero $\phi \in H^1(\mathfrak{n})$ and denote the corresponding leafwise one-form on M by $\tilde{\phi}$. Then $f\tilde{\phi} \in H^1(\mathcal{F}) = H^1(\mathfrak{n})$. Thus there exist $\psi \in H^1(\mathfrak{n})$ and a C^∞-function $g : M \to \mathbb{R}$ such that $f\tilde{\phi} - \tilde{\psi} = d_{\mathcal{F}} g$ where $\tilde{\psi}$ is the leafwise one-form corresponding to ψ. Choose $X \in \mathfrak{n}$ satisfying $\phi(X) \neq 0$. Let $x(t)$ be an integral curve of $\tilde{X}$ where $\tilde{X}$ is the vector field corresponding to X. We have

$$f(x(t))\phi(X) - \psi(X) = \tilde{X}_{x(t)} g = \frac{d}{dt} g(x(t)).$$

By integrating over $[0, T]$, we get $T(f(x(0))\phi(X) - \psi(X)) = g(x(T)) - g(x(0))$. Since g is bounded, $f(x(0))\phi(X) - \psi(X)$ must be zero. Then $f(x(0)) = \psi(X)/\phi(X)$ and f is constant on M.

This completes the proof of Theorem 1. □

4. *A construction of parameter rigid actions*

Let us now construct real-valued cocycle rigid actions of nilpotent groups. For the structure theory of nilpotent Lie groups, see [**2**]. A basis $X_1, \ldots, X_n$ of $\mathfrak{n}$ is called a *strong Malcev basis* if $\operatorname{span}_{\mathbb{R}}\{X_1, \ldots, X_i\}$ is an ideal of $\mathfrak{n}$ for each i. If Γ is a lattice in N, there exists a strong Malcev basis $X_1, \ldots, X_n$ of $\mathfrak{n}$ such that $\Gamma = e^{\mathbb{Z}X_1} \cdots e^{\mathbb{Z}X_n}$. Such a basis is called a *strong Malcev basis strongly based on* Γ. Let Γ and Λ be lattices in N.

Definition 1. Λ is *Diophantine with respect to* Γ if there exists a strong Malcev basis $X_1, \ldots, X_n$ of $\mathfrak{n}$ strongly based on Γ and a strong Malcev basis $Y_1, \ldots, Y_n$ of $\mathfrak{n}$ strongly based on Λ such that $Y_i = \sum_{j=1}^{i} a_{ij} X_j$ for every $1 \leq i \leq n$, where a_{ii} is Diophantine.

Let ρ be the action of Λ on $\Gamma \backslash N$ by right multiplication. First we will prove the following.

THEOREM 4. *If Λ is Diophantine with respect to Γ, then every real-valued C^∞ cocycle $c : \Gamma \backslash N \times \Lambda \to \mathbb{R}$ over ρ is cohomologous to a constant cocycle.*

Proof. Note that X_1 is in the center of $\mathfrak{n}$. Let $\pi : N \to \bar{N} := e^{\mathbb{R}X_1} \backslash N$ be the projection. Since $\Gamma \cap e^{\mathbb{R}X_1} = e^{\mathbb{Z}X_1}$ is a cocompact lattice in $e^{\mathbb{R}X_1}$, $\bar{\Gamma} := \pi(\Gamma) = e^{\mathbb{R}X_1} \backslash \Gamma e^{\mathbb{R}X_1}$ is a cocompact lattice in $\bar{N}$. Let $\bar{\mathfrak{n}} = \mathbb{R}X_1 \backslash \mathfrak{n}$. Then $\bar{X}_2, \ldots, \bar{X}_n$ is a strong Malcev basis of $\bar{\mathfrak{n}}$ strongly based on $\bar{\Gamma}$.

We will see that the naturally induced map $\bar{\pi} : \Gamma \backslash N \to \bar{\Gamma} \backslash \bar{N}$ is a principal S^1-bundle. Indeed,

$$\Gamma \backslash \Gamma e^{\mathbb{R}X_1} \hookrightarrow \Gamma \backslash N \twoheadrightarrow \Gamma e^{\mathbb{R}X_1} \backslash N$$

is a principal $\Gamma \backslash \Gamma e^{\mathbb{R}X_1}$-bundle, and we have

$$\Gamma \backslash \Gamma e^{\mathbb{R}X_1} \simeq \Gamma \cap e^{\mathbb{R}X_1} \backslash e^{\mathbb{R}X_1} = e^{\mathbb{Z}X_1} \backslash e^{\mathbb{R}X_1} \simeq \mathbb{Z} \backslash \mathbb{R}$$

and the following diagram.

$$\begin{tikzcd}
e^{\mathbb{R}X_1} \backslash \Gamma e^{\mathbb{R}X_1} \arrow[r, hook] & e^{\mathbb{R}X_1} \backslash N \arrow[r, two heads] \arrow[d] & \Gamma e^{\mathbb{R}X_1} \backslash N \\
& \bar{\Gamma} \backslash \bar{N} \arrow[ur, "\sim"] &
\end{tikzcd}$$

Since $\Lambda \cap e^{\mathbb{R}X_1} = \Lambda \cap e^{\mathbb{R}Y_1} = e^{\mathbb{Z}Y_1}$ is a cocompact lattice in $e^{\mathbb{R}X_1}$, then $\bar{\Lambda} := \pi(\Lambda)$ is a cocompact lattice in $\bar{N}$. Furthermore, $\bar{Y}_2, \ldots, \bar{Y}_n$ is a strong Malcev basis of $\bar{\mathfrak{n}}$ strongly based on $\bar{\Lambda}$ and $\bar{Y}_i = \sum_{j=2}^{i} a_{ij} \bar{X}_j$ where a_{ii} is Diophantine. Therefore $\bar{\Lambda}$ is Diophantine with respect to $\bar{\Gamma}$.

Since $\bar{\pi}$ is Λ-equivariant, the action ρ of Λ when restricted to $e^{\mathbb{Z}Y_1}$, preserves fibers of $\bar{\pi}$.

Let $z \in \bar{\Gamma}\backslash\bar{N}$. Choose a point Γx in $\bar{\pi}^{-1}(z)$. Then we have a trivialization

$$\iota_{\Gamma x} : \mathbb{Z}\backslash\mathbb{R} \simeq \bar{\pi}^{-1}(z)$$

of $\bar{\pi}^{-1}(z)$ given by $\iota_{\Gamma x}(s) = \Gamma e^{sX_1}x$. Note that if we take another point $\Gamma y \in \bar{\pi}^{-1}(z)$, $\iota_{\Gamma y}^{-1} \circ \iota_{\Gamma x} : \mathbb{Z}\backslash\mathbb{R} \to \mathbb{Z}\backslash\mathbb{R}$ is a rotation.

Let $Y_1 = aX_1$, where a is Diophantine. If we identify $\bar{\pi}^{-1}(z)$ with $\mathbb{Z}\backslash\mathbb{R}$ by $\iota_{\Gamma x}$, then the action of e^{Y_1} on $\mathbb{Z}\backslash\mathbb{R}$ is $s \mapsto s + a$.

Let μ_z be the normalized Haar measure naturally defined on $\bar{\pi}^{-1}(z)$, μ the N-invariant probability measure on $\Gamma\backslash N$ and ν the $\bar{N}$-invariant probability measure on $\bar{\Gamma}\backslash\bar{N}$. For any $f \in C(\Gamma\backslash N)$,

$$\int_{\Gamma\backslash N} f \, d\mu = \int_{\bar{\Gamma}\backslash\bar{N}} \int_{\bar{\pi}^{-1}(z)} f \, d\mu_z \, d\nu. \tag{2}$$

LEMMA 4. *ρ is ergodic with respect to μ.*

Proof. We use induction on n. If $n = 1$, ρ is an irrational rotation on $\mathbb{Z}\backslash\mathbb{R}$, and hence the result is well known. In general, let $f : \Gamma\backslash N \to \mathbb{C}$ be a Λ-invariant L^2-function with $\int_{\Gamma\backslash N} f \, d\mu = 0$. Since the action of $e^{\mathbb{Z}Y_1}$ on $\bar{\pi}^{-1}(z)$ is ergodic, $f|_{\bar{\pi}^{-1}(z)}$ is constant μ_z-almost everywhere. We denote this constant by $g(z)$. Then $g : \bar{\Gamma}\backslash\bar{N} \to \mathbb{C}$ is a $\bar{\Lambda}$-invariant measurable function. By induction, g is constant ν-almost everywhere. By (2), this constant must be zero. Therefore f is zero μ-almost everywhere. □

Let $c : \Gamma\backslash N \times \Lambda \to \mathbb{R}$ be a C^∞-cocycle over ρ. We must show that c is cohomologous to a constant cocycle $c_0 : \Lambda \to \mathbb{R}$ where $c_0(\lambda) := \int_{\Gamma\backslash N} c(x, \lambda) \, d\mu(x)$. Therefore we may assume that $\int_{\Gamma\backslash N} c(x, \lambda) \, d\mu(x) = 0$ for all $\lambda \in \Lambda$, and we will show that c is a coboundary. We prove this by induction on n. When $n = 1$, ρ is a Diophantine rotation on $\mathbb{Z}\backslash\mathbb{R}$, and hence the result is well known.

LEMMA 5. *For all $m \in \mathbb{Z}$,*

$$\int_{\bar{\pi}^{-1}(z)} c(s, e^{mY_1}) \, d\mu_z(s) = 0.$$

Proof. Fix m and put $g(z) = \int_{\bar{\pi}^{-1}(z)} c(s, e^{mY_1}) \, d\mu_z(s)$. For any $\lambda \in \Lambda$, the cocycle equation gives $c(x, \lambda) + c(x\lambda, e^{mY_1}) = c(x, e^{mY_1}) + c(xe^{mY_1}, \lambda)$. By integrating this equation on $\bar{\pi}^{-1}(z)$, we get $g(z\pi(\lambda)) = g(z)$. Since the action of $\bar{\Lambda}$ on $\bar{\Gamma}\backslash\bar{N}$ is ergodic, g is constant. By (2), g must be zero. □

Let $f : \mathbb{Z}\backslash\mathbb{R} \xrightarrow{\iota_{\Gamma x}} \bar{\pi}^{-1}(z) \xrightarrow{c(\cdot, e^{Y_1})} \mathbb{R}$. We define

$$h_z(\iota_{\Gamma x}(s)) = \sum_{k \in \mathbb{Z}\backslash\{0\}} \frac{\hat{f}(k)}{-1 + e^{2\pi i k a}} e^{2\pi i k s}.$$

Then $h_z : \bar{\pi}^{-1}(z) \to \mathbb{R}$ is C^∞, since f is C^∞ and a is Diophantine. By Lemma 5, we have

$$c(\iota_{\Gamma x}(s), e^{Y_1}) = -h_z(\iota_{\Gamma x}(s)) + h_z(\iota_{\Gamma x} e^{Y_1}).$$

If we choose another point $\Gamma e^{s_0 X_1} x \in \bar{\pi}^{-1}(z)$ to define h_z, then

$$\begin{aligned} h_z(\iota_{\Gamma x}(s)) &= h_z(\Gamma e^{sX_1} x) = h_z(\iota_{\Gamma e^{s_0 X_1} x}(s - s_0)) \\ &= \sum_{k \in \mathbb{Z}\setminus\{0\}} \frac{1}{-1 + e^{2\pi i k a}} \int_0^1 c(\Gamma e^{(u+s_0)X_1} x, e^{Y_1}) e^{-2\pi i k u} du \, e^{2\pi i k (s - s_0)} \\ &= \sum_{k \in \mathbb{Z}\setminus\{0\}} \frac{1}{-1 + e^{2\pi i k a}} \int_0^1 f(u + s_0) e^{-2\pi i k u} du \, e^{2\pi i k (s - s_0)} \\ &= \sum_{k \in \mathbb{Z}\setminus\{0\}} \frac{\hat{f}(k)}{-1 + e^{2\pi i k a}} e^{2\pi i k s}, \end{aligned}$$

so h_z is determined only by z. Define $h : \Gamma\backslash N \to \mathbb{R}$ by $h|_{\bar{\pi}^{-1}(z)} = h_z$. Then for all $x \in \Gamma\backslash N$ and $m \in \mathbb{Z}$, $c(x, e^{mY_1}) = -h(x) + h(x e^{mY_1})$.

Let $U \subset \bar{\Gamma}\backslash\bar{N}$ be open and $\sigma : U \to \bar{\pi}^{-1}(U)$ a section of $\bar{\pi}$. Then we have a trivialization $\mathbb{Z}\backslash\mathbb{R} \times U \simeq \bar{\pi}^{-1}(U)$ which sends (s, z) to $\iota_{\sigma(z)}(s) = \Gamma e^{sX_1}\sigma(z)$. Hence

$$h(\iota_{\sigma(z)}(s)) = \sum_{k \in \mathbb{Z}\setminus\{0\}} \frac{1}{-1 + e^{2\pi i k a}} \int_0^1 c(\iota_{\sigma(z)}(u), e^{Y_1}) e^{-2\pi i k u} du \, e^{2\pi i k s}$$

on $\bar{\pi}^{-1}(U)$. The following lemma shows h is C^∞ on $\Gamma\backslash N$.

LEMMA 6. *Let $U \subset \mathbb{R}^n$ be open and let $f : \mathbb{Z}\backslash\mathbb{R} \times U \to \mathbb{R}$ be a C^∞-function. Define*

$$h(s, z) = \sum_{k \in \mathbb{Z}\setminus\{0\}} \frac{1}{-1 + e^{2\pi i k a}} \widehat{f_z}(k) e^{2\pi i k s},$$

where $f_z(u) = f(u, z)$. Then $h : \mathbb{Z}\backslash\mathbb{R} \times U \to \mathbb{R}$ is C^∞.

Proof. Let V be open such that $\bar{V} \subset U$ and $\bar{V}$ is compact. We will show that h is C^∞ on $\mathbb{Z}\backslash\mathbb{R} \times V$. Choose constants $C, \alpha > 0$ such that $|-1 + e^{2\pi i k a}| \geq C|k|^{-\alpha}$ for all $k \in \mathbb{Z}\setminus\{0\}$.

We will first prove that h is continuous. Since for any $m \in \mathbb{Z}_{>0}$,

$$\frac{\partial^m f_z}{\partial s^m}(s) = \sum_{k \in \mathbb{Z}} (2\pi i k)^m \widehat{f_z}(k) e^{2\pi i k s}$$

in $L^2(\mathbb{Z}\backslash\mathbb{R})$,

$$\begin{aligned} \left\| \frac{\partial^m f_z}{\partial s^m} \right\|_2^2 &= \sum_{k \in \mathbb{Z}} |(2\pi i k)^m \widehat{f_z}(k)|^2 \\ &\geq (2\pi)^{2m} |k|^{2m} |\widehat{f_z}(k)|^2 \geq |k|^{2m} |\widehat{f_z}(k)|^2. \end{aligned}$$

Since

$$\left\| \frac{\partial^m f_z}{\partial s^m} \right\|_2 = \left(\int_0^1 \left| \frac{\partial^m}{\partial s^m} f(s, z) \right|^2 ds \right)^{\frac{1}{2}}$$

is continuous in z, there exists $M > 0$ such that $\|\partial^m f_z/\partial s^m\|_2 < M$ for every $z \in \bar{V}$. Hence, for all $k \in \mathbb{Z}$ and $z \in \bar{V}$, $|k|^m|\widehat{f_z}(k)| \le M$. Therefore, for any $z \in \bar{V}$,

$$\sum_{k\in\mathbb{Z}\setminus\{0\}} \left|\frac{1}{-1+e^{2\pi ika}}\widehat{f_z}(k)e^{2\pi iks}\right| \le C^{-1}\sum_{k\in\mathbb{Z}\setminus\{0\}}\frac{1}{|k|^2}|k|^{\alpha+2}|\widehat{f_z}(k)|$$
$$\le C^{-1}M\sum_{k\in\mathbb{Z}\setminus\{0\}}\frac{1}{|k|^2} < \infty.$$

This implies continuity of h on $\mathbb{Z}\backslash\mathbb{R}\times\bar{V}$.

We have

$$\frac{\partial h}{\partial s}(s,z) = \sum_{k\in\mathbb{Z}\setminus\{0\}}\frac{2\pi ik}{-1+e^{2\pi ika}}\widehat{f_z}(k)e^{2\pi iks}.$$

Thus a similar argument shows that $\partial h/\partial s$ is continuous.

Let $z = (z_1, \ldots, z_n)$. For any $z \in \bar{V}$,

$$\left|\frac{\partial}{\partial z_j}\left(\frac{1}{-1+e^{2\pi ika}}\widehat{f_z}(k)e^{2\pi iks}\right)\right| = \left|\frac{1}{-1+e^{2\pi ika}}\widehat{\frac{\partial f}{\partial z_j}(\cdot,z)}(k)e^{2\pi iks}\right|$$
$$\le C^{-1}\frac{1}{|k|^2}|k|^{\alpha+2}\left|\widehat{\frac{\partial f}{\partial z_j}(\cdot,z)}(k)\right|$$
$$\le C^{-1}M'\frac{1}{|k|^2} \in L^1(\mathbb{Z}\setminus\{0\}).$$

Thus

$$\frac{\partial h}{\partial z_j}(s,z) = \sum_{k\in\mathbb{Z}\setminus\{0\}}\frac{1}{-1+e^{2\pi ika}}\widehat{\frac{\partial f}{\partial z_j}(\cdot,z)}(k)e^{2\pi iks}.$$

Hence $\partial h/\partial z_j$ is continuous by an argument similar to those above. For higher derivatives of h, continue this procedure. □

Set $c_1(x,\lambda) = c(x,\lambda) + h(x) - h(x\lambda)$. Then $c_1 : \Gamma\backslash N \times \Lambda \to \mathbb{R}$ is a C^∞-cocycle and $c_1(x, e^{mY_1}) = 0$. Thus for any $\lambda \in \Lambda$, the cocycle equation implies $c_1(x,\lambda) = c_1(xe^{Y_1},\lambda)$. Since the action of $e^{\mathbb{Z}Y_1}$ on $\bar{\pi}^{-1}(z)$ is ergodic, $c_1(x,\lambda)$ is constant on $\bar{\pi}^{-1}(z)$. Therefore we can define a cocycle $\bar{c} : \bar{\Gamma}\backslash\bar{N}\times\bar{\Lambda}\to\mathbb{R}$ by $\bar{c}(\bar{\pi}(x),\pi(\lambda)) = c_1(x,\lambda)$. Indeed, if $\bar{\pi}(x) = \bar{\pi}(y)$ and $\pi(\lambda) = \pi(\lambda')$, then there exists an $m \in \mathbb{Z}$ with $\lambda = e^{mY_1}\lambda'$, so that

$$c_1(x,\lambda) = c_1(x, e^{mY_1}\lambda') = c_1(xe^{mY_1},\lambda') = c_1(y,\lambda').$$

Furthermore,

$$\int_{\bar{\Gamma}\backslash\bar{N}}\bar{c}(x,\pi(\lambda))\,d\nu(z) = \int_{\bar{\Gamma}\backslash\bar{N}}\int_{\bar{\pi}^{-1}(z)}c_1(s,\lambda)\,d\mu_z(s)\,d\nu(z)$$
$$= \int_{\Gamma\backslash N}c_1(x,\lambda)\,d\mu(x) = 0.$$

By induction, there exists a C^∞-function $P : \bar{\Gamma}\backslash\bar{N}\to\mathbb{R}$ such that $\bar{c}(z,\pi(\lambda)) = -P(z) + P(z\pi(\lambda))$. Put $Q = P\circ\bar{\pi}$. Then $c_1(x,\lambda) = \bar{c}(\bar{\pi}(x),\pi(\lambda)) = -Q(x) + Q(x\lambda)$. This proves Theorem 4. □

Proof of Theorem 3. Let $\tilde{\rho} : M \times N \to M$ be the suspension of $\rho : \Gamma\backslash N \times \Lambda \to \Gamma\backslash N$ where $M = \Gamma\backslash N \times_\Lambda N$ is a compact manifold. Then $\tilde{\rho}$ is locally free and let $\mathcal{F}$ be its orbit foliation. We have

$$H^1(\mathcal{F}) \simeq H^1(\Lambda, C^\infty(\Gamma\backslash N))$$

by [**6**], where the right hand side is the first cohomology of the Λ-module $C^\infty(\Gamma\backslash N)$ obtained by ρ. It is easy to prove that $\mathrm{Hom}(\Lambda, \mathbb{R}) \to H^1(\Lambda, C^\infty(\Gamma\backslash N))$ is injective. By Theorem 4,

$$H^1(\Lambda, C^\infty(\Gamma\backslash N)) = \mathrm{Hom}(\Lambda, \mathbb{R}).$$

LEMMA 7. $\dim \mathrm{Hom}(\Lambda, \mathbb{R}) = \dim H^1(\mathfrak{n})$.

Proof. Recall that $[N, N]\backslash\Lambda[N, N]$ is a cocompact lattice in $[N, N]\backslash N$ and that $[\Lambda, \Lambda]\backslash(\Lambda \cap [N, N])$ is finite. Since

$$0 \to [\Lambda, \Lambda]\backslash(\Lambda \cap [N, N]) \to [\Lambda, \Lambda]\backslash\Lambda \to [N, N]\backslash\Lambda[N, N] \to 0$$

is exact, we have

$$\mathrm{rank}\, [\Lambda, \Lambda]\backslash\Lambda = \mathrm{rank}\, [N, N]\backslash\Lambda[N, N] = \dim [N, N]\backslash N.$$

Thus

$$\begin{aligned} \dim \mathrm{Hom}(\Lambda, \mathbb{R}) &= \dim \mathrm{Hom}([\Lambda, \Lambda]\backslash\Lambda, \mathbb{R}) \\ &= \mathrm{rank}\, [\Lambda, \Lambda]\backslash\Lambda \\ &= \dim [N, N]\backslash N \\ &= \dim \mathrm{Hom}_{\mathbb{R}}([\mathfrak{n}, \mathfrak{n}]\backslash\mathfrak{n}, \mathbb{R}) \\ &= \dim H^1(\mathfrak{n}). \end{aligned}$$ □

Therefore we obtain

$$H^1(\mathcal{F}) = H^1(\mathfrak{n}).$$

This proves Theorem 3. □

5. *Existence of Diophantine lattices*

Let $\mathfrak{n}_{\mathbb{Q}}$ be a rational structure of $\mathfrak{n}$. We construct Diophantine lattices when $\mathfrak{n}_{\mathbb{Q}}$ admits a graduation. Namely, we assume that $\mathfrak{n}_{\mathbb{Q}}$ has a sequence V_i of $\mathbb{Q}$-subspaces such that $\mathfrak{n}_{\mathbb{Q}} = \bigoplus_{i=1}^k V_i$ and $[V_i, V_j] \subset V_{i+j}$. Let $X_1, \ldots, X_n$ be a $\mathbb{Q}$-basis of $\mathfrak{n}_{\mathbb{Q}}$ such that $X_1, \ldots, X_{i_1} \in V_k, X_{i_1+1}, \ldots, X_{i_2} \in V_{k-1}, \ldots, X_{i_{k-1}+1}, \ldots, X_n \in V_1$. Then $X_1, \ldots, X_n$ is a strong Malcev basis of $\mathfrak{n}$ with rational structure constants. Multiplying $X_1, \ldots, X_n$ by an integer if necessary, we may assume that $\Gamma := e^{\mathbb{Z}X_1} \cdots e^{\mathbb{Z}X_n}$ is a cocompact lattice in N. Let α be a root of an irreducible polynomial of degree $k+1$ over $\mathbb{Q}$. Since $\alpha, \alpha^2, \ldots, \alpha^k$ are irrational algebraic numbers, they are Diophantine. If we define a linear map $\varphi : \mathfrak{n} \to \mathfrak{n}$ by $\varphi(X) = \alpha^i X$ for $X \in V_i \otimes \mathbb{R}$, then φ is an automorphism of Lie algebra $\mathfrak{n}$. Put $Y_i = \varphi(X_i)$. Then $Y_1, \ldots, Y_n$ is a strong Malcev basis of $\mathfrak{n}$ strongly based on $\Lambda := e^{\mathbb{Z}Y_1} \cdots e^{\mathbb{Z}Y_n}$. Thus Λ is Diophantine with respect to Γ.

Acknowledgement. The author would like to thank his advisor, Masayuki Asaoka, for helpful comments.

REFERENCES

[1] M. Asaoka. Deformation of locally free actions and the leafwise cohomology. arXiv:1012.2946.

[2] L. Corwin and F. P. Greenleaf. *Representations of Nilpotent Lie Groups and Their Applications. Part 1:Basic Theory and Examples (Cambridge Studies in Advanced Mathematics, 18)*. Cambridge University Press, Cambridge, 1990.

[3] A. Katok and R. J. Spatzier. First cohomology of Anosov actions of higher rank abelian groups and applications to rigidity. *Publ. Math. Inst. Hautes Études Sci.* **79** (1994), 131–156.

[4] S. Matsumoto and Y. Mitsumatsu. Leafwise cohomology and rigidity of certain Lie group actions. *Ergod. Th. & Dynam. Sys.* **23** (2003), 1839–1866.

[5] D. Mieczkowski. The first cohomology of parabolic actions for some higher-rank abelian groups and representation theory. *J. Mod. Dyn.* **1** (2007), 61–92.

[6] M. S. Pereira and N. M. dos Santos. On the cohomology of foliated bundles. *Proyecciones* **21**(2) (2002), 175–197.

[7] F. A. Ramírez. Cocycles over higher-rank abelian actions on quotients of semisimple Lie groups. *J. Mod. Dyn.* **3** (2009), 335–357.

[8] N. M. dos Santos. Parameter rigid actions of the Heisenberg groups. *Ergod. Th. & Dynam. Sys.* **27** (2007), 1719–1735.